ON THE FAMILY OF ALL TOPOLOGIES ON A SET

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I. Introduction.

A collection $\boldsymbol{T}$ of subsets of a non-empty set X satisfying the following conditions:

(O_1) the union of any subcollection of $\boldsymbol{T}$ belongs to $\boldsymbol{T}$;

(O_2) the intersection of a finite subcollection of $\boldsymbol{T}$ belongs to $\boldsymbol{T}$;

(O_3) $\phi \in \boldsymbol{T}$, $X \in \boldsymbol{T}$;

is a topology on X. This is the most general way to define a topology on X. Under this definition, the non-empty family $\boldsymbol{F}$ of all topologies on X is determined. In general the familly $\boldsymbol{F}$ containes at least two topologies, namely, the trivial topology and the discrete topology.

This paper is devoted to find, among $\boldsymbol{F}$, some special topologies which have some interesting properties and relations.

II. Topology-valued mappings.

Let $\boldsymbol{C}$ be the collection of all subsets of a non-empty set X. A subcollection $\boldsymbol{B} \subset \boldsymbol{C}$ need not be a topology on X. The idea of the subbase (see [2] pp. 44-48; [3] pp. 46-48; [4] pp. 99-102) is to generate $\boldsymbol{B}$ to become a topology on X. To do this two operators F, F* are introduced here. For any $\boldsymbol{B} \subset \boldsymbol{C}$,

$$\text{F:} \quad F(\boldsymbol{B}) = \{T: T = \bigcup_{g \in G} \bigcap_{i \in I} B,\ B \in \boldsymbol{B}\},$$

where $\bigcup_{g \in G} \bigcap_{i \in I} B = \bigcup_{g \in G} (\bigcap B_i)_g$, I is any finite index set, and G is any arbitrary index set.

It is well known that $F(\boldsymbol{B}) = \boldsymbol{T}$ is a topology and the weakest topology containing the members of $\boldsymbol{B}$ as open sets on the topological space $\{X, \boldsymbol{T}\}$ (see [2] pp. 44-47; [3] p. 48). Hence the operator F can be considered as a topology-valued mapping which maps all subcollections of $\boldsymbol{C}$ onto the family $\boldsymbol{F}$ of all topologies on X.

The operator F* is defined as

$$\text{F*:} \quad F^*(\boldsymbol{B}) = \{T: T = \bigcap_{g \in G} \bigcup_{i \in I} B,\ B \in \boldsymbol{B}\}.$$

Unfortunately, $F^*(\boldsymbol{B})$ is not always a topology on X, for example, let $\boldsymbol{T}$ be the cofinite topology on an infinite set X, i. e., whenever $T \in \boldsymbol{T}$, (X-T) is a finite subset of X. Then the complement $\boldsymbol{T}_c$ of $\boldsymbol{T}$ consists of its members with all finite subsets of X. It is obvious that an arbitrary union of its members is not always a finite subset of X. This does not meet the condition (O_1). However, the following theorems will be helpful.

Theorem 1. $F_c(\boldsymbol{B})=F^*(\boldsymbol{B}_c)$ for any $\boldsymbol{B}\subset \boldsymbol{C}$.

Note that the symbols used in this theorem, and also in the sequel, are defined as follows:

(i) $\boldsymbol{B}_c$ is the complement collection of $\boldsymbol{B}$, i.e.,

$\boldsymbol{B}_c=\{B_c: B_c=(X-B),\ B\in \boldsymbol{B}\}$,

(ii) $F_c(\boldsymbol{B})=[F(\boldsymbol{B})]_c$

Proof: The following diagram is useful in the proof of this theorem.

$$\begin{array}{ccc} \boldsymbol{B} & \xrightarrow{\quad F \quad} & F(\boldsymbol{B}) \\ \updownarrow c & & \updownarrow c \\ \boldsymbol{B}_c & \xrightarrow{\quad F^* \quad} & F^*(\boldsymbol{B}_c) \end{array}$$

Let $T\in F_c(\boldsymbol{B})$. Then, by the De Morgan formulas,

$$T\Longleftrightarrow(X-\bigcup_{g\in G}\bigcap_{i\in I}B)\Longleftrightarrow\bigcap_{g\in G}(X-\bigcap_{i\in I}B)\Longleftrightarrow\bigcap_{g\in G}\bigcup_{i\in I}(X-B)\Longleftrightarrow\bigcap_{g\in G}\bigcup_{i\in I}B_c\Longleftrightarrow T^*\in F^*(\boldsymbol{B}_c).$$

Corollary. $F_c^*(\boldsymbol{B})$ is a topology and the weakest topology such that all members of $\boldsymbol{B}$ are closed sets in the topological space $\{X, F^*_c(\boldsymbol{B})\}$.

This is a convenient way for one to find the weakest topology on X having the desired property mentioned in the corollary above.

Theorem 2. $F(\boldsymbol{B})=F^*(\boldsymbol{B})$ for any finite subcollection $\boldsymbol{B}\subset \boldsymbol{C}$.

Proof: Although the operators $\cup$, $\cap$ are not commutative, yet the members of the two collections $F(\boldsymbol{B})$, $F^*(\boldsymbol{B})$ are one to one equals. This is due to the formulas

$$(\bigcup_{i\in I}A_i)\cap(\bigcup_{j\in J}B_j)=\bigcup_{\substack{i\in I\\ j\in J}}(A_i\cap B_j),$$

$$(\bigcap_{i\in I}A_j)\cup(\bigcap_{j\in J}B_j)=\bigcap_{\substack{i\in I\\ j\in J}}(A_i\cup B_j).$$

III. Dual topologies.

It has been pointed out, in Sec. II, that $F^*(\boldsymbol{B})$ need not be a topology on X if $\boldsymbol{B}$ is not finite, and hence neither the complement $\boldsymbol{T}_c$ of a topology $\boldsymbol{T}$ on X. (see Theorem 1). So it is interesting to characterize a topology $\boldsymbol{T}$ whose complement $\boldsymbol{T}_c$ is also a topology on X.

Definition. A topology $\boldsymbol{T}$ on X is a dual topology if the intersection of any subcollection of $\boldsymbol{T}$ belongs to $\boldsymbol{T}$.

Theorem 3. For any topology $\boldsymbol{T}$ on X, $\boldsymbol{T}$ is a dual topology iff $\boldsymbol{T}_c$ is a topology on X.

Proof: Suppose that $\boldsymbol{T}$ is a dual topology. From the formulas

$$\bigcap_{g\in G}\{(X-T_g): T_g\in \boldsymbol{T},\ g\in G\}=X-\bigcup_{g\in G}\{T_g: T_g\in T,\ g\in G\},$$

$$\bigcup_{g\in G}\{(X-T_g): T_g\in \boldsymbol{T},\ g\in G\}=X-\bigcap_{g\in G}\{T_g: T_g\in \boldsymbol{T},\ g\in G\},$$

clearly, $\boldsymbol{T}_c$ is a topology on X.

Conversely, if $\boldsymbol{T}$ and $\boldsymbol{T}_c$ are both topologies on X, there exists a subbase $\boldsymbol{B}$ of $\boldsymbol{T}$ such that

$F(\boldsymbol{B})=\boldsymbol{T}$ and $F^*(\boldsymbol{B}_c)=\boldsymbol{T}_c$ by the diagram in Sec. II.

It is enough to prove that $\mathcal{T}_c$ is a dual topology. Since $\mathcal{T}_c$ is a topology, it is sufficient to prove that the intersection of any subcollection of $\mathcal{T}_c$ belongs to $\mathcal{T}_c$. But the collection $\mathcal{T}_c$ is

$$\mathcal{T}_c = F^*(\mathcal{B}_c) = \{T_c: T_c = \bigcap_{g \in G} \bigcup_{i \in I} B_c,\ B_c \in \mathcal{B}_c\}.$$

and by the generalized commutative and associative laws for intersections,

$$\bigcap_{h \in H} T_c = \bigcap_{h \in H} (\bigcap_{g \in G} \bigcup_{i \in I} B_c) = \bigcap_{k \in K} \bigcup_{i \in I} B_c$$

holds for any arbitrary index set H. Hence $\mathcal{T}_c$ is a dual topology on X, and therefore the proof is complete.

The consequences below follow immediately from the theorem.

Corollary 1. If $\mathcal{T}$ is a dual topology, then $\mathcal{T}_c$ is also a dual topology.

Corollary 2. If $\mathcal{T}$ and $\mathcal{T}_c$ are both topologies on X, then they are both dual topologies. They form a pair of mutual dual topologies on X.

IV. Filer topologies.

It is well known that the intersection of two topologies on X is a topology on X, however, the union of two topologies on X is not necessarily a topology on X. To be sure to obtain their union as a topology on X, some specific topologies are needed. In order to characterize these topologies, it is convenient to use the properties of the filter below (see [2] pp. 150-151; [3] p. 83).

A *filter $\mathcal{L}$ on a set X is a collection of non-empty subsets of X such that*

($\mathcal{L}_1$) *the intersection of two members of $\mathcal{L}$ belongs to $\mathcal{L}$,*

($\mathcal{L}_2$) if $B \in \mathcal{L}$ and $B \subset D \subset X$, then $D \in \mathcal{L}$.

Definition. A topology $\mathcal{T}$ on X is a filter topology on X iff $(\mathcal{T}\text{-}\phi)$ is a filter on X.

It is evident that for any filter $\mathcal{L}$, $\phi \cup \mathcal{L}$ is a topology and hence a filter topology.

Theorem 4. If one of the two mutual dual topolgies on X is a filter topology, then their union is a topology on X.

Proof: Let $\mathcal{T}$ and $\mathcal{T}_c$ be two mutual dual topologies on X, and one of them a filter topology, say $\mathcal{T}$. Then, by (L_2), any union of the members of $\mathcal{T} \cup \mathcal{T}_c$ belongs to $\mathcal{T} \cup \mathcal{T}_c$.

Next, let T_1 and T_2 be any two members of $\mathcal{T} \cup \mathcal{T}_c$. It can be easily seen that $T_1 \cap T_2 \in \mathcal{T} \cup \mathcal{T}_c$ whenever both T_1 and T_2 belong to $\mathcal{T}$ or $\mathcal{T}_c$. Now in the case that $T_1 \in \mathcal{T}$, $T_2 \in \mathcal{T}_c$, the fact that $T_1 \cap T_2 \in \mathcal{T} \cup \mathcal{T}_c$ is by no means trivial. Suppose that $T_1 \cap T_2 \notin \mathcal{T}$. Then it is sufficient to prove that $T_1 \cap T_2 \in \mathcal{T}_c$. Since the topology $\mathcal{T}$ is the complement collection of $\mathcal{T}_c$, so T_1 and $(X\text{-}T_2)$ belong to $\mathcal{T}$ and hence $T_1 \cap (X\text{-}T_2) = T_1 - (T_1 \cap T_2) \in \mathcal{T}$. Further, because $\mathcal{T}$ is a filter topology, it follows that $X\text{-}(T_1 \cap T_2) \in \mathcal{T}$. Therefore $T_1 \cap T_2 \in \mathcal{T}_c$. The proof will become complete by induction.

Because the topology $\mathcal{T} \cup \mathcal{T}_c$ is a union of two mutual complement collections, every member of this topology is both open and closed. So that Theorem 4 shows a way to construct a topology, other than the trivial topology and the discrete topology, whose members are both open and closed.

Example. A pair of two mutual dual topologies on the set X={a,b,c,d} are

$$\mathcal{T} = \{\phi, (ab), (abc), (abd), X\},$$

$T_c = \{X, (cd), d, c, \phi\}$.

Clearly, T is a filter topology and hence their union

$$T \cup T_c = \{\phi, c, d, (ab), (cd), (abc), (abd), X\}$$

is a topology on X.

References

1. Paul R. Halmos, Naive Set Theory (1960).
2. D. Bushaw, Elements of General Topology (1963).
3. John L. Kelly, General Topology (1955).
4. George F. Simmons, Introduction to Topology and Modern Analysis (1963).

集合上之拓撲族

解 萬 臣

在某一集合 X 上定義拓撲 (Topology) 之方法很多，一般常以開集合 (Open Set) 之性質定義之。今將X上所有之子集合 (Subset) 組成一聚集 (Collection)，此聚集中任一子聚集 (Subcollection) 在上述之定義下即可被確定其是否爲一 X 上之拓撲。在 X 上所有之拓撲形成一拓撲族 (The family of all Topologies on X)。在此拓撲族中任一拓撲之餘集聚集 (Complement Collection) (由每一元素之餘集合組成)未必爲一拓撲；又在此拓撲族中任二拓撲其併和 (Union) 通常不爲一拓撲；本文將討論具有何種性質之拓撲其餘集聚集仍爲一拓撲；又具有何種性質之二拓撲其併合仍爲一拓撲。